

# Non-Gaussian density fluctuations in the Dean-Kawasaki equation

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Computing analytically the  $n$ -point density correlations in systems of interacting particles is a long-standing problem of statistical physics with a broad range of applications, from the interpretation of scattering experiments in simple liquids, to the understanding of their collective dynamics. For Brownian particles, i.e., with overdamped Langevin dynamics, the microscopic density obeys a stochastic evolution equation, known as the Dean-Kawasaki equation. In spite of the importance of this equation, its complexity makes it very difficult to analyze the statistics of the microscopic density beyond simple Gaussian approximations. In this work, resorting to a path-integral description of the stochastic dynamics and relying on a saddle-point analysis in the limit of high density and weak interactions between the particles, we go beyond the usual linearization of the Dean-Kawasaki equation, and we compute exactly the three- and four-point density correlation functions. This result opens the way to using the Dean-Kawasaki equation beyond the simple Gaussian treatments, and it could find applications to understand many fluctuation-related effects in soft and active matter systems.

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## I. INTRODUCTION

Characterizing  $n$ -point density correlations in systems of interacting particles is a central problem of statistical physics. For instance, measuring two-point density correlations of liquids is generally the simplest insight into their structure and dynamics. Such observables have motivated a significant amount of theoretical work and the development of numerical methods, rooted in classical, equilibrium statistical mechanics [1–3]. In addition, they found their applications in the interpretation of neutron, x-ray, or light scattering experiments, which have become central tools to analyze liquids, and more generally soft matter [4–7]. More complex observables have also attracted attention: three-point (or “triplet”) correlations have been studied in order to improve the description of liquids beyond simple two-body approaches, and to get better insight into their structure and dynamics [8–14].

From an analytical perspective, the explicit calculation of  $n$ -point density correlations is a notoriously difficult problem, and it typically requires approximations. For instance, to compute static two-point functions, the well-known Percus-Yevick, hypernetted chain, or mean spherical closures have been successful [3,15]. Their dynamical counterpart can be estimated using schemes such as mode-coupling theory (MCT) [16], which remain valid (up to a certain extent) even for very slow dynamics, and in which high-order density correlations were characterized [17]. However, in spite of their predictive power, such theories almost always rely on numerical evaluations, and fully analytical results are scarce, independently of the underlying dynamics of the system (Newtonian or Langevin).

We will consider the ubiquitous model of Brownian particles which interact via pair potentials—this level of description is particularly relevant to describe colloidal particles, macromolecular or polymeric fluids, and ions. The positions of the particles,  $\mathbf{r}_1(t), \dots, \mathbf{r}_N(t)$ , obey overdamped Langevin equations, while the empirical or microscopic density, defined as  $\rho(\mathbf{x}, t) \equiv \sum_{i=1}^N \delta(\mathbf{x} - \mathbf{r}_i(t))$ , obeys the Dean-Kawasaki (DK) equation [18,19]. The latter is particularly difficult to analyze and has only been studied in specific regimes. First, in the absence of interactions, the statistics of  $\rho$  can be fully characterized as Poissonian (either from direct calculation or from a field-theoretical formulation), as one would expect from simple physical considerations [20]. Second, a one-loop renormalized treatment of the DK equation highlighted its connections with standard MCT theory [21]—this approach is nonetheless not fully consistent, as it is an expansion around a Gaussian theory, and not around the case of noninteracting particles. Finally, another commonly used strategy consists in expanding the DK equation in the limit of high density and weak interactions [22–24]. At leading order, the resulting “linearized” equation, which has been extensively used to study a variety of fluctuation-related phenomena in soft and active matter [25–30], is by nature limited to describing the Gaussian fluctuations of the stochastic field  $\rho$ . The characterization of non-Gaussian fluctuations beyond these limiting regimes has been addressed numerically [31], but remains an open problem from an analytical perspective.

Given the broad range of domains where the DK equation has been studied, from purely theoretical or computational considerations to applications in the physics of soft and active matter for the interpretation of fluctuation-induced phenomena [32], this appears as a prominent line of research, where explicit analytical solutions are still scarce. In this article, resorting to a path-integral formulation and to a

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perturbative expansion similar to that employed in macroscopic fluctuation theory, we go beyond previous treatments of the DK equation and we compute the three- and four-point correlation functions of the microscopic density of interacting Brownian particles. We obtain a simple and fully explicit analytical expression that is confronted to numerical simulations. This constitutes the first analytical characterization of non-Gaussian fluctuations in the DK equation with interactions.

## II. THE DEAN-KAWASAKI EQUATION

Consider a  $d$ -dimensional suspension of  $N$  particles interacting via some pair potential  $V(\mathbf{r})$ , and obeying overdamped dynamics. We denote by  $\mu$  their bare mobility, and  $D$  their bare diffusion coefficient, which are related through the fluctuation-dissipation relation  $D = \mu k_B T$ , where  $k_B$  is Boltzmann's constant and  $T$  is the temperature. Let us start from the set of Langevin equations

$$\frac{d\mathbf{r}_\alpha}{dt} = -\mu \sum_{\beta=1}^N \nabla V(\mathbf{r}_\alpha - \mathbf{r}_\beta) + \sqrt{2D} \boldsymbol{\xi}_\alpha(t), \quad (1)$$

where  $\boldsymbol{\xi}_\alpha(t)$  are Gaussian white noises, of average zero and variances  $\langle \boldsymbol{\xi}_{\alpha,i}(t) \boldsymbol{\xi}_{\beta,j}(t') \rangle = \delta_{\alpha\beta} \delta_{ij} \delta(t - t')$ . The evolution equation of the density of particles, defined as  $\rho(\mathbf{x}, t) \equiv \sum_{\alpha=1}^N \delta(\mathbf{x} - \mathbf{r}_\alpha(t))$ , reads

$$\begin{aligned} \frac{\partial}{\partial t} \rho(\mathbf{x}, t) = & \nabla \cdot [\boldsymbol{\xi}(\mathbf{x}, t) \sqrt{2D\rho(\mathbf{x}, t)}] + D \nabla^2 \rho(\mathbf{x}, t) \\ & + \mu \nabla \cdot \left[ \rho(\mathbf{x}, t) \int d\mathbf{y} \rho(\mathbf{y}, t) \nabla V(\mathbf{x} - \mathbf{y}) \right], \end{aligned} \quad (2)$$

where  $\boldsymbol{\xi}(\mathbf{x}, t)$  is a Gaussian white noise of average zero and variance  $\langle \xi_i(\mathbf{x}, t) \xi_j(\mathbf{x}', t') \rangle = \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$ . This exact evolution equation, usually called the Dean-Kawasaki equation, was derived from phenomenological considerations by the latter [18], and later obtained using stochastic calculus by the former [19]. This equation has progressively become an important object of study, as it encompasses under a compact form the  $N$ -body dynamics described by the coupled overdamped Langevin equations given in Eq. (1). Its complexity lies in its nonlinearity and in the multiplicative noise term, which makes its direct resolution impossible as such.

A simple and straightforward treatment of the DK equation consists in linearizing the stochastic density  $\rho$  around a constant uniform state  $\rho_0 = N/\mathcal{V}$  (where  $\mathcal{V} = L^d$  is the volume of the system), i.e., writing  $\rho(\mathbf{x}, t) = \rho_0 + \sqrt{\rho_0} \phi(\mathbf{x}, t)$ , and considering the joint limit limit  $\phi \ll \sqrt{\rho_0}$  and  $V \rightarrow 0$  with constant  $\rho_0 V$  (i.e., the limit of weak interactions and high density, which can be understood as a mean-field-like treatment) [22–24]. At leading order, this typically leads to a linear equation obeyed by the perturbation  $\phi$  (Appendix A), which can be solved in Fourier space [33] to yield  $\tilde{\phi}(\mathbf{k}, \omega) = \sqrt{2D} \tilde{\eta}(\mathbf{k}, \omega) / [i\omega + \Omega(\mathbf{k})]$ , where  $\Omega(\mathbf{k})$  is homogeneous to an inverse time and is defined as  $\Omega(\mathbf{k}) = Dk^2 + \mu \rho_0 k^2 \tilde{V}(\mathbf{k})$ , and where  $\tilde{\eta}$  is a Gaussian white noise of zero average and variance  $\langle \tilde{\eta}(\mathbf{k}, t) \tilde{\eta}(\mathbf{k}', t') \rangle = (2\pi)^d k^2 \delta(\mathbf{k} + \mathbf{k}') \delta(t - t')$ . Therefore, the perturbation  $\phi$  is clearly a Gaussian field, and this approximation does not allow the characterization of the non-Gaussian behavior of the field  $\rho$ . More precisely, within this linearization, the  $n$ -point connected correlation

functions of  $\rho$ , i.e., the cumulants of the density, trivially vanish for  $n \geq 3$ .

## III. PATH-INTEGRAL FORMULATION

To go beyond the Gaussian approximation, we follow the path-integral formulation of the stochastic dynamics that was initially proposed by Martin-Siggia-Rose [34] and Janssen [35]. Its present application is similar to the path-integral treatment of the equation of fluctuating hydrodynamics in other contexts [36–40]. The Dean-Kawasaki equation [Eq. (2)] can be rewritten under the form  $\partial_t \rho = -\nabla \cdot \mathbf{J}$ , with the stochastic current  $\mathbf{J} = -D \nabla \rho - \sqrt{2D\rho} \boldsymbol{\xi} - \mu \rho (\rho * \nabla V)$ , where we define the convolution operator  $(f * g)(\mathbf{x}) = \int d\mathbf{y} f(\mathbf{y}) g(\mathbf{x} - \mathbf{y})$ . Since  $\boldsymbol{\xi}$  is a unit Gaussian white noise, its probability distribution functional reads  $p[\boldsymbol{\xi}] \propto \exp[-\frac{1}{2} \int_0^T dt' \int d\mathbf{x}' \boldsymbol{\xi}(\mathbf{x}', t')^2]$ , up to a normalization constant, where  $T$  is the typical observation time of the trajectory. The probability to observe a given trajectory  $\rho$  knowing the initial configuration  $\rho(\mathbf{x}, 0)$  then reads

$$\mathcal{P}[\rho | \rho(\mathbf{x}, 0)] \sim \int \mathcal{D}\boldsymbol{\xi} \delta(\partial_t \rho + \nabla \cdot \mathbf{J}) e^{[-\frac{1}{2} \int_0^T dt' \int d\mathbf{x}' \boldsymbol{\xi}(\mathbf{x}', t')^2]} \quad (3)$$

up to a normalization prefactor. Note that the DK equation (2) is a stochastic differential equation with multiplicative noise, which is interpreted here in the Itô way [41,42]. A consequence is that the Jacobian of the transformation to a path-integral is constant, and is absorbed in the normalization [43,44].

Next, we use the Fourier representation of the  $\delta$ -distribution, which reads, for any functional  $\psi[\rho]$ :  $\delta(\psi[\rho]) = \int \mathcal{D}\hat{\rho} \exp\{-\int_0^T dt \int d\mathbf{x} \hat{\rho}(\mathbf{x}, t) \psi[\rho(\mathbf{x}, t)]\}$ , where  $\hat{\rho}$  is an auxiliary field. With this representation, it is now possible to rewrite the integral as  $\mathcal{P}[\rho | \rho(\mathbf{x}, 0)] \sim \int \mathcal{D}\hat{\rho} \mathcal{D}\boldsymbol{\xi} \exp\{-\int_0^T dt \int d\mathbf{x} [\hat{\rho}(\partial_t \rho + \nabla \cdot \mathbf{J}) + \frac{\boldsymbol{\xi}^2}{2}]\}$ . Recalling the definition of  $\mathbf{J}$ , one gets

$$\begin{aligned} \mathcal{P}[\rho | \rho(\mathbf{x}, 0)] \\ \sim \int \mathcal{D}\hat{\rho} e^{-\int_0^T dt d\mathbf{x} \hat{\rho}(\partial_t \rho - D \nabla^2 \rho - \mu \nabla[\rho(\rho * \nabla V)])} G[\rho, \hat{\rho}] \end{aligned} \quad (4)$$

where  $G[\rho, \hat{\rho}] = \int \mathcal{D}\boldsymbol{\xi} e^{-\int_0^T dt d\mathbf{x} [\frac{\boldsymbol{\xi}^2}{2} + \sqrt{2D\rho} \nabla \hat{\rho} \cdot \boldsymbol{\xi}]}$ , i.e.,  $G[\rho, \hat{\rho}] = e^{\int_0^T dt d\mathbf{x} D \rho (\nabla \hat{\rho})^2}$ . Finally, performing integration by parts with respect to  $\mathbf{x}$  yields  $\mathcal{P}[\rho | \rho(\mathbf{x}, 0)] \sim \int \mathcal{D}\hat{\rho} \exp\{-\int_0^T dt \int d\mathbf{x} S[\rho, \hat{\rho}]\}$  with the action

$$\begin{aligned} S[\rho, \hat{\rho}] \equiv & \hat{\rho} \partial_t \rho - D \rho (\nabla \hat{\rho})^2 + D (\nabla \hat{\rho}) \cdot (\nabla \rho) \\ & + \mu \rho (\nabla \hat{\rho}) \cdot (\rho * \nabla V). \end{aligned} \quad (5)$$

Incorporating the weight of the initial condition yields  $P[\rho] \sim \int \mathcal{D}\hat{\rho} \exp\{-S[\rho, \hat{\rho}]\}$ , with the generalized action  $S[\rho, \hat{\rho}] = -\ln\{P_0[\rho(\mathbf{x}, t=0)]\} + \int_0^T dt \int d\mathbf{x} S[\rho, \hat{\rho}]$ , where  $P_0$  is the initial distribution of the density.

## IV. $n$ -POINT CORRELATION FUNCTIONS

We now aim to calculate the  $n$ -point density correlation functions defined in real space as  $C_n(\mathbf{X}_1, \dots, \mathbf{X}_n) = \langle \prod_{k=1}^n \rho(\mathbf{X}_k) \rangle_c$ , where the index “c” indicates a connected

correlation function, and where we introduce the shorthand notation  $\mathbf{X} \equiv (\mathbf{x}, t)$ . They can be computed through the successive functional derivatives of the cumulant-generating functional  $\mu[\lambda] = \ln \langle \exp[\int d\mathbf{X} \lambda(\mathbf{X}) \rho(\mathbf{X})] \rangle$  (where the angular brackets denote average with respect to the distribution  $P$ ), with respect to  $\lambda(\mathbf{X}_1), \dots, \lambda(\mathbf{X}_n)$ , and where we ultimately set  $\lambda \equiv 0$ . Equivalently, for  $n \geq 2$ , the connected correlation functions can be computed as

$$C_n(\mathbf{X}_1, \dots, \mathbf{X}_n) = \frac{\delta^{n-1} \langle \rho(\mathbf{X}_1) \rangle_\lambda}{\delta \lambda(\mathbf{X}_2) \cdots \delta \lambda(\mathbf{X}_n)} \Big|_{\lambda \equiv 0}, \quad (6)$$

where the average  $\langle \cdot \rangle_\lambda$  is taken with respect to the tilted action  $S_\lambda[\rho, \hat{\rho}] = - \int d\mathbf{X} \lambda(\mathbf{X}) \rho(\mathbf{X}) + S[\rho, \hat{\rho}]$ , i.e.,  $\langle \cdot \rangle_\lambda = \frac{\int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \cdot \exp(-S_\lambda[\rho, \hat{\rho}])}{\int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \exp(-S[\rho, \hat{\rho}])}$ .

It is generally not possible to obtain explicit expressions when computing averages with respect to the action  $S_\lambda$ , as the integrals cannot be performed easily. However, one can make progress in the joint limit of high density ( $\rho_0 \rightarrow \infty$ ) and weak interactions ( $V \rightarrow 0$ ), with the product of  $\rho_0 V$  being of order 1—this is the limit that was discussed in Sec. II and that leads to the “linearized” Dean-Kawasaki equation. In this limit, the tilted action can be rewritten as  $S_\lambda[\rho, \hat{\rho}] = \rho_0 \tilde{S}_\lambda[\bar{\rho}, \hat{\rho}]$ , where  $\bar{\rho}$  is such that  $\rho = \rho_0 \bar{\rho}$ , and where  $\tilde{S}_\lambda[\bar{\rho}, \hat{\rho}]$  is of order 1 and independent of  $\rho_0$ . Applying a saddle-point method, it becomes clear that the average with the tilted action  $S_\lambda$  is dominated by the most probable path, i.e., the path  $(\rho, \hat{\rho})$  which minimizes the action. This approach is reminiscent of the strategy that is at the heart of macroscopic fluctuation theory (MFT) [45–47], where saddle point approximations are typically justified by hydrodynamic limits (i.e., long times and large distances).

We denote the path that minimizes the action by  $(q, p)$ , and we study small variations around it, i.e., we set  $\rho(\mathbf{X}) = q(\mathbf{X}) + \delta\rho(\mathbf{X})$  and  $\hat{\rho}(\mathbf{X}) = p(\mathbf{X}) + \delta\hat{\rho}(\mathbf{X})$ . Defining  $\delta S_\lambda \equiv S_\lambda(\rho, \hat{\rho}) - S_\lambda(q, p)$  and performing variational calculus, we find that the path of least action, for which  $\delta S_\lambda$  cancels, obeys the following equations (Appendix B):

$$\partial_t q = D \nabla^2 q - 2D \nabla \cdot (q \nabla p) + \mu \nabla \cdot [q(q * \nabla V)], \quad (7)$$

$$\begin{aligned} \partial_t p &= -D(\nabla p)^2 - D \nabla^2 p + \mu(\nabla p) \cdot (q * \nabla V) \\ &\quad - \mu(q \nabla p) * \nabla V - \lambda, \end{aligned} \quad (8)$$

where the Lagrange multiplier  $\lambda$  acts as a source in the equation for  $p$ . These “bulk” equations are completed by initial and final conditions on the values of the fields  $q$  and  $p$ , which are given in Appendix B. Although Eqs. (7) and (8) resemble the MFT equations that were studied for one-dimensional diffusive systems [36–40, 48–52], they are valid here in  $d$  spatial dimensions, and the pair interactions are encoded explicitly in pair potential  $V$ , rather than in the macroscopic transport coefficients (diffusivity and conductivity) that appear in MFT.

To summarize, the connected density correlation functions are computed using Eq. (6), i.e., through the successive functional derivatives of  $\langle \rho(\mathbf{X}_1) \rangle_\lambda$ , which is approximated as  $q(\mathbf{X}_1)$  under the saddle-point approximation. The function  $q$  is obtained as the solution of the set of Eqs. (7) and (8). Importantly, since the determination of  $C_n$  through Eq. (6) involves taking  $(n-1)$  functional derivatives with respect to  $\lambda$  and then taking  $\lambda \equiv 0$ , one only needs to solve for  $q$  at

order  $\lambda^{n-1}$ . To this end, we will introduce the following series expansion:  $f(\mathbf{X}) = \sum_{n=0}^{\infty} f_n(\mathbf{X})$  (for  $f = p$  or  $q$ ), where  $f_n$  is of order  $n$  in  $\lambda$ , and we solve Eqs. (7) and (8) order by order.

Before solving Eqs. (7) and (8), one needs to specify additional hypotheses and conditions: (i) First, at order 0 in  $\lambda$ , the solution of Eqs. (7) and (8) must follow the average evolution of the system. This implies  $p_0(\mathbf{x}, t) = 0$  for all  $\mathbf{x}, t$ , and we choose to expand the dynamics around the profile  $q_0(\mathbf{x}, t) = \rho_0$  in order to be consistent with the usual expansion of the Dean-Kawasaki equation around a constant, uniform state, as discussed in Sec. II. (ii) Second, we choose to solve these equations in the limit where the system is equilibrated, i.e., when the initial conditions can be sent to the infinite past [20]. More rigorously, when computing the  $n$ th correlation functions, it means that we take the limit  $t_1, \dots, t_n \rightarrow \infty$ , with finite differences between any two times  $t_i - t_j$ . In practice, we use Fourier transformations for both space and time to solve Eqs. (7) and (8). We emphasize that, in principle, Eqs. (7) and (8) must be solved for a given initial condition. One can either consider quenched initial conditions [i.e., with a fixed, deterministic initial density profile  $\rho_q$ , implying  $P_0[\rho] = \delta(\rho - \rho_q)$ ], or annealed (i.e., with a density profile drawn from its equilibrium distribution, implying  $P_0[\rho] = e^{-\beta \mathcal{F}[\rho]}$ , where  $\mathcal{F}$  is the free energy of the system) [36, 37, 48, 53]. The consequences of these choices on the initial and final values of the fields  $q$  and  $p$  are recalled in Eq. (B4). (iii) Finally, we emphasize that this approach will yield exact results in the limit where  $\rho_0$  is very large and where the typical interaction potential  $V$  is very small, with the product  $\rho_0 V$  being finite.

As a side remark, we show in Appendix C that, starting from Eqs. (7) and (8), taking  $V = 0$ , and using a Cole-Hopf transformation [37], one retrieves the exact results for the  $n$ -point correlation functions of the noninteracting Brownian gas [20].

## V. EXPLICIT RESULTS

It will be convenient to compute the connected correlation functions in Fourier space through the relation:

$$\begin{aligned} \tilde{C}_n(\mathbf{K}_1, \dots, \mathbf{K}_n) &= \left\langle \prod_{i=1}^n \tilde{\rho}(\mathbf{K}_i) \right\rangle_c \\ &= (2\pi)^{(n-1)(d+1)} \frac{\delta^{n-1} \tilde{q}(\mathbf{K}_1)}{\delta \tilde{\lambda}(-\mathbf{K}_2) \cdots \delta \tilde{\lambda}(-\mathbf{K}_n)} \Big|_{\tilde{\lambda} \equiv 0}, \end{aligned} \quad (9)$$

where we use the shorthand notation  $\mathbf{K} = (\mathbf{k}, \omega)$ . It will also be convenient to introduce the quantity  $\psi_n$ , which is such that

$$\tilde{C}_n(\mathbf{K}_1, \dots, \mathbf{K}_n) = \rho_0 (2\pi)^{d+1} \psi_n(\mathbf{K}_1, \dots, \mathbf{K}_n) \delta \left( \sum_i \mathbf{K}_i \right). \quad (10)$$

### A. Two-point correlations

Expanding both  $p$  and  $q$  in powers of  $\lambda$ , and at order 1 in the perturbation, it is straightforward to show that Eqs. (7) and (8) yield  $\tilde{q}_1(\mathbf{K}) = 2D\rho k^2 \mathcal{G}(\mathbf{K}) \tilde{p}_1(\mathbf{K})$  and  $\tilde{p}_1(\mathbf{K}) = \mathcal{G}(\mathbf{K}) * \tilde{\lambda}(\mathbf{K})$ ,

where we introduce the Green's function  $\mathcal{G}(\mathbf{K}) = [i\omega + \Omega(\mathbf{k})]^{-1} = [i\omega + Dk^2 + \mu\rho_0 k^2 \tilde{V}(\mathbf{k})]^{-1}$ . Using Eq. (9), one obtains

$$\langle \tilde{\rho}(\mathbf{K}_1) \tilde{\rho}(\mathbf{K}_2) \rangle_c = \frac{2D\rho_0 k_1^2 (2\pi)^{d+1}}{\Omega(\mathbf{k}_1)^2 + \omega_1^2} \delta(\mathbf{k}_1 + \mathbf{k}_2) \delta(\omega_1 + \omega_2). \quad (11)$$

One can check that these are the same correlations as the ones that may be computed within the linearization of the Dean-Kawasaki equation around the homogeneous density  $\rho_0$ , as described above. Inverting the Fourier transforms with respect to time, Eq. (11) yields the following expression for the intermediate scattering function, defined for a finite-size system [54] as  $\mathcal{F}_2(k, t) = \frac{1}{N} \langle \tilde{\rho}(\mathbf{k}, t) \tilde{\rho}(-\mathbf{k}, 0) \rangle = \frac{Dk^2}{\Omega(k)} e^{-Dk^2[1+\tilde{v}(k)]t}$ , where we introduce the dimensionless potential  $\tilde{v}(k) \equiv \rho_0 \tilde{V}(k)/k_B T$ . As a consequence, the two-point structure factor that is obtained within this perturbative expansion is  $\mathcal{S}(k) = \mathcal{F}_2(k, t=0) = [1 + \tilde{v}(k)]^{-1}$ , which coincides with the structure factor that is typically computed within the random phase approximation [3,23,55,56].

### B. Three-point correlations

We now go one step further and compute the deviation from the Gaussian behavior, which is encoded in higher-order correlation functions. To compute three-point correlation functions, we need to solve Eqs. (7) and (8) at order 2 in  $\lambda$ , which read

$$\begin{aligned} \partial_t q_2 &= D\nabla^2 q_2 + \mu\rho_0 \nabla \cdot (\nabla V * q_2) - 2D\rho_0 \nabla^2 p_2 + Q_2, \\ \partial_t p_2 &= -D\nabla^2 p_2 - \rho_0 \mu \nabla V * \nabla p_2 + P_2. \end{aligned} \quad (12)$$

The source terms  $Q_2$  and  $P_2$  are explicit in terms of  $q_1$  and  $p_1$ , which both have been calculated when we studied the equations at order 1 in  $\lambda$ :

$$\begin{aligned} Q_2 &= -2D\nabla \cdot (q_1 \nabla p_1) + \mu \nabla \cdot [q_1 (\nabla V * q_1)], \\ P_2 &= -D(\nabla p_1)^2 + \mu (\nabla p_1) \cdot (\nabla V * q_1) - \mu (q_1 \nabla p_1 * \nabla V). \end{aligned} \quad (13)$$

In Fourier space, the function  $q_2$ , which is needed to compute three-point correlation functions through Eq. (9), is formally obtained as  $\tilde{q}_2(\mathbf{K}) = -2D\rho_0 k^2 |\mathcal{G}(\mathbf{K})|^2 \tilde{P}_2(\mathbf{k}, \omega) + \mathcal{G}(\mathbf{K}) \tilde{Q}_2(\mathbf{k}, \omega)$ . Using Eq. (9) for  $n=3$ , and computing the functional derivatives of  $\tilde{q}_2$  with respect to  $\tilde{\lambda}$ , we get an explicit expression for the three-point correlation function in Fourier space:  $\tilde{C}_3(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3) = \rho_0 (2\pi)^{d+1} \delta(\mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3) \psi_3(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3)$ , with

$$\begin{aligned} \psi_3(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3) &= \Gamma_3(\mathbf{K}_1, \mathbf{K}_2) \psi_2(\mathbf{K}_3) + \Gamma_3(\mathbf{K}_1, \mathbf{K}_3) \psi_2(\mathbf{K}_2) \\ &\quad + \Gamma_3(\mathbf{K}_2, \mathbf{K}_3) \psi_2(\mathbf{K}_1), \end{aligned} \quad (14)$$

where  $\psi_2(\mathbf{K})$  is a shorthand for  $\psi_2(-\mathbf{K}, \mathbf{K})$ , which is defined from the two-point correlation functions [Eq. (11)] as  $\psi_2(\mathbf{K}, \mathbf{K}') = -2D(\mathbf{k} \cdot \mathbf{k}') \mathcal{G}(\mathbf{K}) \mathcal{G}(\mathbf{K}')$ , and  $\Gamma_3(\mathbf{K}, \mathbf{K}')$  reads

$$\begin{aligned} \Gamma_3(\mathbf{K}, \mathbf{K}') &= \psi_2(\mathbf{K}, \mathbf{K}') + \mu \mathbf{k} \cdot \mathbf{k}' [\mathcal{G}(\mathbf{K}) \tilde{V}(\mathbf{K}') \psi_2(\mathbf{K}') \\ &\quad + \mathbf{K} \leftrightarrow \mathbf{K}']. \end{aligned} \quad (15)$$

The Fourier transforms with respect to the frequencies  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  can all be inverted explicitly, yielding an explicit but lengthy expression for  $\langle \tilde{\rho}(\mathbf{k}_1, t_1) \tilde{\rho}(\mathbf{k}_2, t_2) \tilde{\rho}(\mathbf{k}_3, t_3) \rangle_c$  for

$t_1 < t_2 < t_3$  (Appendix D). To get a simpler expression and to limit the number of variables, we consider the particular situation where  $t_1 = 0$  and  $t_2 = t_3 = t > 0$ . We get

$$\begin{aligned} &\langle \tilde{\rho}(\mathbf{k}_1, 0) \tilde{\rho}(\mathbf{k}_2, t) \tilde{\rho}(\mathbf{k}_3, t) \rangle_c \\ &= \mathcal{S}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \left\{ e^{-\Omega(\mathbf{k}_1)t} + [1 + \tilde{v}(\mathbf{k}_1)] \right. \\ &\quad \times \left. \frac{\mathbf{k}_1 \cdot [\mathbf{k}_2 \tilde{v}(\mathbf{k}_2) + \mathbf{k}_3 \tilde{v}(\mathbf{k}_3)] (e^{-[\Omega(\mathbf{k}_2) + \Omega(\mathbf{k}_3)]t} - e^{-\Omega(\mathbf{k}_1)t})}{[\Omega(\mathbf{k}_1) - \Omega(\mathbf{k}_2) - \Omega(\mathbf{k}_3)]} \right\}, \end{aligned} \quad (16)$$

where we defined the static three-point structure factor as

$$\mathcal{S}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{(2\pi)^d \rho_0 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)}{[1 + \tilde{v}(\mathbf{k}_1)][1 + \tilde{v}(\mathbf{k}_2)][1 + \tilde{v}(\mathbf{k}_3)]}. \quad (17)$$

Equation (16), which is exact, is the central result of this paper, and several comments follow: (i) In the absence of interactions ( $V=0$ ), one retrieves the expression of the normalized correlation function that can be derived straightforwardly when the positions of the particles  $\mathbf{r}_i(t)$  are independent Wiener processes [20], and which reads, for a finite-size system,  $\mathcal{F}_3(\mathbf{k}, \mathbf{k}', t) \equiv \frac{1}{N} \langle \tilde{\rho}(\mathbf{k}, t) \tilde{\rho}(\mathbf{k}', t) \tilde{\rho}(-\mathbf{k} - \mathbf{k}', 0) \rangle = e^{-D(\mathbf{k} + \mathbf{k}')^2 t}$ . (ii) It is clear that the third-cumulant of the density is different from the second one, meaning that the distribution of  $\rho$  is generally non-Poissonian (apart from the special case of noninteracting particles [20]). (iii) The typical relaxation time of the dynamics, which reads  $\tau(\mathbf{k}) = \Omega(\mathbf{k})^{-1} = \{Dk^2[1 + \rho_0 \tilde{V}(\mathbf{k})/k_B T]\}^{-1}$  in this limit, controls the relaxation of the three-point correlations. As one could expect, it decreases for increasing density and interactions. (iv) The expression obtained for the static three-point structure factor [Eq. (17)] coincides with the expression that is usually obtained within the “convolution” or Kirkwood approximation [57,58]. It consists in writing  $\mathcal{S}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \simeq \mathcal{S}(\mathbf{k}_1) \mathcal{S}(\mathbf{k}_2) \mathcal{S}(\mathbf{k}_3)$ , where  $\mathcal{S}(\mathbf{k})$  is the two-point structure factor that can be proven by writing the three-point extension of the Ornstein-Zernike approximation, and by setting the three-point direct correlation to zero [3,12]. (v) It can be proven from the analytical expression given in Eq. (16) that  $\mathcal{F}_3(\mathbf{k}_1, \mathbf{k}_2, t)$  is nonmonotonous and always has a negative minimum. This is observed in the plots shown in Fig. 1, where the analytical expression is confronted by results from numerical simulations (see Appendix F—note that, when compared with numerics, the theory is only expected to be exact asymptotically for high density and weak interactions). This means that, at long enough times, the density distribution is negatively skewed. This is in contrast with the case of noninteracting particles, where the density is always positively skewed. Finally, and as expected, we observe that the agreement improves as we approach the joint limit  $\rho_0 \rightarrow +\infty$  and  $V \rightarrow 0$  while keeping the product  $\rho_0 V$  fixed.

### C. Four-point correlations

We finally go one order further and derive the expression of four-point connected correlation functions. This requires the resolution of the equations of motion (7) and (8) at



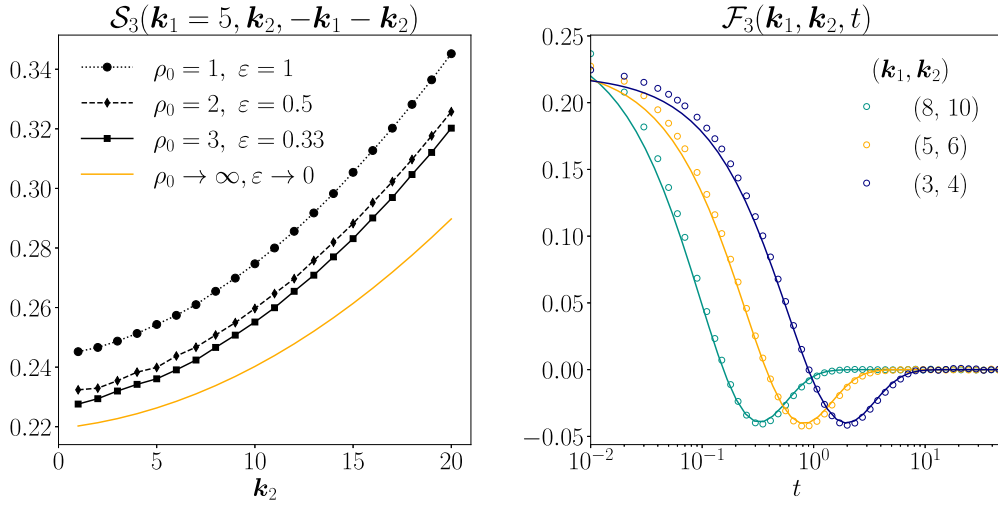


FIG. 1. Results from numerical simulations (symbols) performed in  $d = 1$ , with  $N = 100$  particles interacting via harmonic repulsion:  $V(x) = \varepsilon(1 - x/\sigma)^2\Theta(\sigma - x)$  ( $\sigma = 1, \varepsilon = 0.5$ ), compared with the analytical results (solid lines), Eqs. (16) and (17) for the left and right panel, respectively. See Appendix F for details on numerical simulations. Left: Three-point static structure factor  $\mathcal{S}_3(k_1, k_2, -k_1 - k_2)$  for  $k_1 = 5$  and as a function of  $k_2$ . Right: Three-point correlation function  $\mathcal{F}_3(k_1, k_2, t) = \frac{1}{N} \langle \tilde{\rho}(k_1, t) \tilde{\rho}(k_2, t) \tilde{\rho}(-k_1 - k_2, 0) \rangle$  as a function of time, for  $(k_1, k_2) = (5, 6)$ , and for different values of the overall density  $\rho_0$ . Wave vectors are expressed in units of  $2\pi/L$ .

order 3 in  $\lambda$ , which read

$$\begin{aligned} \partial_t q_3 &= D \nabla^2 q_3 + \mu \rho_0 \nabla \cdot [(q_3 * \nabla V)] - 2D \rho_0 \nabla^2 p_3 + Q_3, \\ \partial_t p_3 &= -D \nabla^2 p_3 - \mu \rho_0 (\nabla p_3 * \nabla V) + P_3. \end{aligned} \quad (18)$$

Here, the source terms  $P_3$  and  $Q_3$  depend on the previously determined fields  $p_1, p_2, q_1$ , and  $q_2$ :

$$\begin{aligned} Q_3 &= -2D \nabla \cdot (q_1 \nabla p_2 + q_2 \nabla p_1) \\ &\quad + \mu \nabla \cdot [q_1 (q_2 * \nabla V) + q_2 (q_1 * \nabla V)], \end{aligned} \quad (19)$$

$$\begin{aligned} P_3 &= -2D \nabla p_1 \cdot \nabla p_2 + \mu \nabla p_1 \cdot (q_2 * \nabla V) \\ &\quad + \mu \nabla p_2 \cdot (q_1 * \nabla V) - \mu [q_1 \nabla p_2 + q_2 \nabla p_1] * \nabla V. \end{aligned} \quad (20)$$

In Fourier space, the set of equations (18) can be solved for both  $\tilde{p}_3$  and  $\tilde{q}_3$ :

$$\begin{aligned} \tilde{q}_3(\mathbf{k}) &= -2D \rho_0 k^2 |\mathcal{G}(\mathbf{k})|^2 \tilde{P}_3(\mathbf{k}) + \mathcal{G}(\mathbf{k}) \tilde{Q}_3(\mathbf{k}), \\ \tilde{p}_3(\mathbf{k}) &= -\mathcal{G}(\mathbf{k})^* \tilde{P}_3(\mathbf{k}). \end{aligned} \quad (21)$$

It should be emphasized that this procedure closely parallels the one used to compute the three-point correlation functions. The structure of Eq. (18), which contains linear terms in  $p_3$  and  $q_3$  and source terms dependent on lower-order  $p_i$  and  $q_i$ , is general and expected to hold at arbitrary order. In practice, the main difficulty lies in the rapidly increasing complexity of the source terms  $P_3$  and  $Q_3$ , which quickly become intractable.

From Eq. (21), one can compute the functional derivatives of  $\tilde{q}_3$  with respect to  $\tilde{\lambda}$  to obtain an explicit expression for

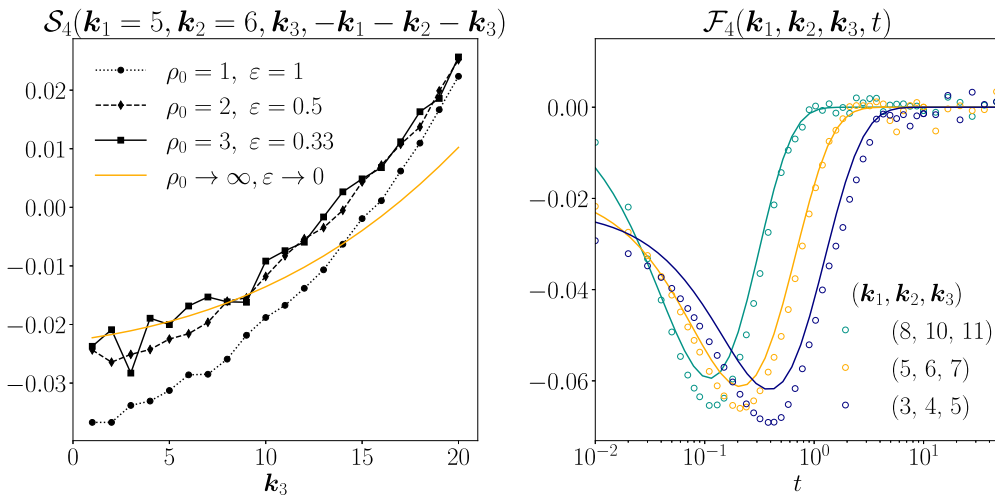


FIG. 2. Results from numerical simulations (symbols) performed in  $d = 1$ , compared with the analytical results (see the caption of Fig. 1 and Appendix F for details on numerical simulations). Left: Four-point static structure factor  $\mathcal{S}_4$  for  $k_1 = 5, k_2 = 6$ , and as a function of  $k_3$ . Right: Four-point correlation function  $\mathcal{F}_4$  as defined in Eq. (22) as a function of time, for  $(k_1, k_2) = (5, 6)$ , and for different values of the overall density  $\rho_0$ .

the fourth cumulant  $\tilde{C}_4$ , i.e., applying Eq. (9) for  $n = 4$ . Its expression is lengthy but given in a rather compact form in Appendix E. A Python notebook is made available: it contains the explicit formula for an arbitrary potential  $V$ , together with its Fourier inversion with respect to the frequencies  $\omega_i$  [59].

Finally, this expression has been validated by computing numerically the four-point scattering function  $\mathcal{F}_4$ , defined as

$$\mathcal{F}_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, t) = \frac{1}{N} \langle \tilde{\rho}(\mathbf{k}_1, t) \tilde{\rho}(\mathbf{k}_2, t) \tilde{\rho}(\mathbf{k}_3, 0) \tilde{\rho}(-\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3, 0) \rangle \quad (22)$$

and the static structure factor  $\mathcal{S}_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \mathcal{F}_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, t = 0)$ , in the one-dimensional setting already considered in the previous section. The theoretical and numerical agreement is presented in Fig. 2. Again, like in the results shown in Fig. 1, we observe that the agreement improves as we approach the joint limit  $\rho_0 \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  with a fixed product. The four-point correlations exhibit a nontrivial nonmonotonic time dependence and may be negative, in contrast with the noninteracting case where it is a positive monotonically decreasing function of time.

## VI. CONCLUSION AND OUTLOOK

In this work, we provided an analytical description of non-Gaussian density fluctuations in the DK equation. Using a path-integral formulation and macroscopic fluctuation theory, we computed three-point density correlation functions for interacting Brownian particles in the regime of high density and weak interactions. These results, and more generally this methodology, establish the potential of the DK framework to capture higher-order fluctuation phenomena in soft and active matter systems. Looking forward, we aim to compute the full large deviation function of the density, i.e., for arbitrary functional  $\lambda$ , in the fashion of the full solution of the MFT equations that were obtained in the context of one-dimensional lattice gases [38,50]. Finally, extensions to multiple coupled stochastic fields will be crucial to take further the description of electrolytes within the DK framework [27,60–63], and more specifically their non-Gaussian fluctuations [64,65].

## ACKNOWLEDGMENTS

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## DATA AVAILABILITY

The data that support the findings of this article are openly available [59].

## APPENDIX A: REMINDER: LINEARIZATION OF THE DEAN-KAWASAKI EQUATION

In this Appendix, we recall the results that are obtained when the Dean-Kawasaki equation [Eq. (2) in the main text] is linearized around a homogeneous state. The Dean-Kawasaki equation is nonlinear in the density  $\rho$ . It can be linearized in the limit of small fluctuations around a constant uniform value  $\rho_0$  [22–24]:  $\rho(\mathbf{x}, t) = \rho_0 + \sqrt{\rho_0} \phi(\mathbf{x}, t)$ . Equation (2) then becomes, after having divided both sides by  $\sqrt{\rho_0}$ ,

$$\begin{aligned} \partial_t \phi(\mathbf{x}, t) = & D \nabla^2 \phi(\mathbf{x}, t) + \mu \rho_0 \nabla \cdot [\phi * \nabla V](\mathbf{x}, t) \\ & + \mu \sqrt{\rho_0} \nabla \cdot [\phi(\phi * \nabla V)] + \sqrt{2D} \nabla \cdot \xi(\mathbf{x}, t) \\ & + \frac{D}{2\sqrt{\rho_0}} \nabla \cdot [\xi(\mathbf{x}, t) \phi(\mathbf{x}, t)], \end{aligned} \quad (A1)$$

with the convolution operator  $(V * \phi)(\mathbf{x}, t) \equiv \int d\mathbf{y} V(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y}, t)$ . In the limit  $\phi \ll \sqrt{\rho_0}$  and  $V \rightarrow 0$  with  $\rho_0 V = O(1)$ , two terms may be neglected: the term proportional to  $\phi^2$ , and the multiplicative noise term. This yields

$$\begin{aligned} \partial_t \phi(\mathbf{x}, t) = & D \nabla^2 \phi(\mathbf{x}, t) + \rho_0 \mu \nabla^2 [(V * \phi)(\mathbf{x}, t)] \\ & + \sqrt{2D} \nabla \cdot \xi(\mathbf{x}, t). \end{aligned} \quad (A2)$$

In Fourier space, the equation reads

$$\begin{aligned} i\omega \tilde{\phi}(\mathbf{k}, \omega) = & -Dk^2 \tilde{\phi}(\mathbf{k}, \omega) - \mu \rho_0 k^2 \tilde{V}(\mathbf{k}) \tilde{\phi}(\mathbf{k}, \omega) \\ & + \sqrt{2D} \tilde{\eta}(\mathbf{k}, \omega), \end{aligned} \quad (A3)$$

where the (scalar) noise  $\tilde{\eta}$  has zero average and the following variance:

$$\langle \tilde{\eta}(\mathbf{k}, \omega) \tilde{\eta}(\mathbf{k}', \omega') \rangle = (2\pi)^{d+1} k^2 \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega'). \quad (A4)$$

Solving for  $\tilde{\phi}$ , one gets

$$\tilde{\phi}(\mathbf{k}, \omega) = \frac{\sqrt{2D}}{(i\omega + Dk^2) + \mu \rho_0 k^2 \tilde{V}(\mathbf{k})} \tilde{\eta}(\mathbf{k}, \omega). \quad (A5)$$

This is the result given in the main text.

We then get the two-point, two-time correlation function:

$$\begin{aligned} \langle \tilde{\phi}(\mathbf{k}, \omega) \tilde{\phi}(\mathbf{k}', \omega') \rangle &= \frac{2D}{\omega^2 + [Dk^2 + \mu \rho_0 k^2 \tilde{V}(\mathbf{k})]^2} (2\pi)^{d+1} k^2 \delta(\mathbf{k} + \mathbf{k}') \\ &\times \delta(\omega + \omega'). \end{aligned} \quad (A6)$$

The Fourier transform with respect to time can be inverted to yield (for  $t > 0$ )

$$\begin{aligned} \langle \tilde{\phi}(\mathbf{k}, t) \tilde{\phi}(\mathbf{k}', 0) \rangle &= (2\pi)^d \delta(\mathbf{k} + \mathbf{k}') \frac{D}{D + \mu \rho_0 \tilde{V}(\mathbf{k})} e^{-[Dk^2 + \mu \rho_0 k^2 \tilde{V}(\mathbf{k})]t}. \end{aligned} \quad (A7)$$

At equal time, one simply gets

$$\langle \tilde{\phi}(\mathbf{k}, 0) \tilde{\phi}(\mathbf{k}', 0) \rangle = (2\pi)^d \delta(\mathbf{k} + \mathbf{k}') \frac{1}{1 + \rho_0 \tilde{V}(\mathbf{k})/k_B T}, \quad (A8)$$

which coincides with the expression of the structure factor that can be computed from the random phase approximations (RPA) [3]. This is the result that is also obtained from the saddle-point treatment of the path-integral formulation of the

stochastic dynamics, at order 1 in  $\lambda$ , and that is given in the main text.

Finally, given that  $\phi$  is Gaussian, all the connected correlation functions of order  $n \geq 3$  are zero within this approximation.

## APPENDIX B: PATH OF LEAST ACTION

In this Appendix, we give details on the derivation of the equations obeyed by the fields  $p$  and  $q$  that minimize the action  $S_\lambda$ , i.e., Eqs. (7) and (8) in the main text. We define  $\delta\rho$  and  $\delta\hat{\rho}$ , the fluctuations around the optimal path  $(q, p)$ ,

$$\rho(\mathbf{x}, t) = q(\mathbf{x}, t) + \delta\rho(\mathbf{x}, t), \quad (\text{B1})$$

$$\hat{\rho}(\mathbf{x}, t) = p(\mathbf{x}, t) + \delta\hat{\rho}(\mathbf{x}, t), \quad (\text{B2})$$

with  $\delta\rho, \delta\hat{\rho} \ll q, p$ . Defining  $\delta S_\lambda \equiv S_\lambda(\rho, \hat{\rho}) - S_\lambda(q, p)$ , we find, at leading order in  $\delta\rho$  and  $\delta\hat{\rho}$ ,

$$\begin{aligned} \delta S_\lambda \simeq & \int d\mathbf{x} \delta\rho(\mathbf{x}, 0) \left[ -\frac{\delta \ln P_0[q]}{\delta q(\mathbf{x}, 0)} - p(\mathbf{x}, 0) \right] + \int d\mathbf{x} \delta\rho(\mathbf{x}, T) p(\mathbf{x}, T) \\ & + \int_0^T dt \int d\mathbf{x} \delta\rho(\mathbf{x}, t) \{ -\partial_t p - D(\nabla p)^2 - D\nabla^2 p + \mu(\nabla p) \cdot (q * \nabla V) - \mu(q\nabla p) * \nabla V - \lambda \} \\ & + \int_0^T dt \int d\mathbf{x} \delta\hat{\rho}(\mathbf{x}, t) \{ \partial_t q - D\nabla^2 q + 2D\nabla \cdot (q\nabla p) - \mu\nabla \cdot [q(q * \nabla V)] \}. \end{aligned} \quad (\text{B3})$$

By definition of  $(q, p)$ , the variation  $\delta S$  given in Eq. (B3) should vanish for all perturbations  $\delta\rho$  and  $\delta\hat{\rho}$ . The first term in the first line of Eq. (B3) yields two possible conditions, depending on whether one chooses the “annealed” or the “quenched” setting. In the “annealed” setting,  $\delta\rho(\mathbf{x}, 0)$  is not specified *a priori*, so the terms between brackets should vanish. On the other hand, in the “quenched” setting, one has  $\delta\rho(\mathbf{x}, 0) = 0$ . In particular, if we consider the initial configuration to be uniform, we get  $q(\mathbf{x}, 0) = \rho_0$ . The second term in the first line of Eq. (B3) yields the other boundary condition:  $p(\mathbf{x}, T) = 0$ . In summary, the boundary conditions read

$$(\text{annealed}) \begin{cases} p(\mathbf{x}, 0) = -\frac{\delta \ln P_0[q]}{\delta q(\mathbf{x}, 0)}, \\ p(\mathbf{x}, T) = 0, \end{cases} \quad (\text{quenched}) \begin{cases} q(\mathbf{x}, 0) = \rho_0, \\ p(\mathbf{x}, T) = 0. \end{cases} \quad (\text{B4})$$

The last two lines in Eq. (B3) give the bulk equations:

$$\partial_t q = D\nabla^2 q - 2D\nabla \cdot (q\nabla p) + \mu\nabla \cdot [q(q * \nabla V)], \quad (\text{B5})$$

$$\partial_t p = -D(\nabla p)^2 - D\nabla^2 p + \mu(\nabla p) \cdot (q * \nabla V) - \mu(q\nabla p) * \nabla V - \lambda, \quad (\text{B6})$$

whose solution with the boundary conditions specified in Eq. (B4) gives the path of least action and the most probable realization of the stochastic density obeying the Dean-Kawasaki equation. These correspond to Eqs. (7) and (8) in the main text.

## APPENDIX C: NONINTERACTING PARTICLES

In this Appendix, we show that all connected correlation functions can be computed for noninteracting particles, thus retrieving the result of [20]. The starting point is the equations of least action (7) and (8), with the potential  $V$  set to zero. The equations of least action read

$$\partial_t q = D\nabla^2 q - 2D\nabla \cdot [q\nabla p], \quad \partial_t p = -D(\nabla p)^2 - D\nabla^2 p - \lambda.$$

This set of coupled PDEs can be decoupled using the Cole-Hopf transformation [37],  $(Q, P) = (qe^{-p}, e^p)$ , such that  $Q$  (resp.  $P$ ) satisfies a diffusion (resp. antidiusion) equation with a spatially and temporally dependent growth term  $\lambda$ :

$$\partial_t Q = D\nabla^2 Q + \lambda Q, \quad \partial_t P = -D\nabla^2 P - \lambda P.$$

These equations can be solved order by order in  $\lambda$  by going in  $\mathbf{k}$ -space for  $Q_n$  and  $P_n$ , which are of order  $n$  in  $\lambda$ . Using the boundary conditions  $Q_0(\mathbf{x}, t) = \rho_0$ ,  $P_0(\mathbf{x}, t) = 1$ , which correspond to the solutions of the noiseless system, one finds that for  $n > 0$ ,

$$\tilde{Q}_n(\mathbf{k}, t) = \int_0^t ds e^{-Dk^2(t-s)} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \tilde{\lambda}(\mathbf{k} - \mathbf{q}, s) \tilde{Q}_{n-1}(\mathbf{q}, s), \quad (\text{C1})$$

$$\tilde{P}_n(\mathbf{k}, t) = - \int_t^T ds e^{+Dk^2(t-s)} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \tilde{\lambda}(\mathbf{k} - \mathbf{q}, s) \tilde{P}_{n-1}(\mathbf{q}, s), \quad (\text{C2})$$

and  $\tilde{Q}_0(\mathbf{k}, t) = (2\pi)^d \rho_0 \delta(\mathbf{k})$ ,  $\tilde{P}_0(\mathbf{k}, t) = (2\pi)^d \delta(\mathbf{k})$ . These expressions for the fields  $\tilde{Q}$  and  $\tilde{P}$  allow one to compute the  $n$ -point connected correlation function  $\tilde{C}_n$  by adapting Eq. (9) from the main text:

$$\begin{aligned} \tilde{C}_n(\mathbf{k}_1, \dots, \mathbf{k}_n; t_1, \dots, t_n) &= (2\pi)^{(n-1)d} \frac{\delta^{n-1} \tilde{q}(\mathbf{k}_n, t_n)}{\delta \tilde{\lambda}(-\mathbf{k}_1, t_1) \cdots \delta \tilde{\lambda}(-\mathbf{k}_{n-1}, t_{n-1})} \Big|_{\tilde{\lambda}=0} \\ &= \delta_1 \cdots \delta_{n-1} \left[ \sum_{i=1}^n \int \frac{d^d \mathbf{q}}{(2\pi)^d} \tilde{P}_i(\mathbf{k}_n - \mathbf{q}, t_n) \tilde{Q}_{n-i}(\mathbf{q}, t_n) \right], \end{aligned}$$

where  $\delta_i(\cdots) \equiv (2\pi)^d \delta(\cdots) / \delta \tilde{\lambda}(-\mathbf{k}_i, t_i)$ . Without loss of generality, choosing the time ordering  $0 < t_1 < \cdots < t_n$  ensures that all functional derivatives act only on  $\tilde{Q}$ , since  $\tilde{P}_i(\mathbf{k}, t_n)$  [Eq. (C2)], viewed as a functional of  $\tilde{\lambda}$ , is independent of  $\tilde{\lambda}(\mathbf{k}, t)$  for  $t < t_n$ . This crucial simplification implies that only the  $(n-1)$ th-order term  $\tilde{Q}_{n-1}$  in the expansion is needed to compute the  $n$ -point connected correlation function, greatly reducing the complexity of the calculation to

$$\tilde{C}_n(\mathbf{k}_1, \dots, \mathbf{k}_n; t_1, \dots, t_n) = (2\pi)^{(n-1)d} \frac{\delta^{n-1} \tilde{Q}_{n-1}(\mathbf{k}_n, t_n)}{\delta \tilde{\lambda}(-\mathbf{k}_{n-1}, t_{n-1}) \cdots \delta \tilde{\lambda}(-\mathbf{k}_1, t_1)}.$$

The situation is now the opposite of that for  $\tilde{P}$ . Indeed, for  $\tilde{Q}_{n-2}(\mathbf{q}, s)$  [see Eq. (C1)], the integrals involved span times from 0 to  $t_{n-2}$ , implying that functional derivatives with respect to  $\tilde{\lambda}(-\mathbf{k}_i, t_i)$  vanish for any  $t_i > t_{n-2}$ . Consequently, the only nonvanishing contributions arise from the combination  $\delta_{n-1}[\tilde{\lambda}(\mathbf{k}_n - \mathbf{q}, s)] \delta_{n-2} \cdots \delta_1[\tilde{Q}_{n-1}(\mathbf{q}, s)]$ . This structure leads to a recursive formula for the connected  $n$ -point correlation functions of the perfect Brownian gas:

$$\tilde{C}_n(\mathbf{k}_1, \dots, \mathbf{k}_n; t_1, \dots, t_n) = e^{-Dk_n^2(t_n - t_{n-1})} \tilde{C}_{n-1}(\mathbf{k}_1, \dots, \mathbf{k}_{n-1} + \mathbf{k}_n; t_1, \dots, t_{n-1}). \quad (\text{C3})$$

The two-point correlation function, which sets the initial condition for the recursive formula, reads

$$\tilde{C}_2(\mathbf{k}_1, \mathbf{k}_2; t_1, t_2) = \rho_0 (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2) \exp[D\mathbf{k}_1 \cdot \mathbf{k}_2(t_2 - t_1)].$$

Using this recursive formula, one can derive an explicit expression for the  $n$ -point connected correlation function:

$$\langle \tilde{\rho}(\mathbf{k}_1, t_1) \cdots \tilde{\rho}(\mathbf{k}_n, t_n) \rangle_c = (2\pi)^d \delta\left(\sum_i \mathbf{k}_i\right) \rho_0 \exp\left\{D \sum_{i < j} \mathbf{k}_i \cdot \mathbf{k}_j (t_j - t_i)\right\}. \quad (\text{C4})$$

Although our derivation was done in the limit  $\rho_0 \rightarrow \infty$ , the final result is valid at arbitrary density [20].

As a concluding remark regarding the noninteracting Brownian gas, we note that the recursion relation (C3) can be Fourier-transformed back to real space. This yields the expression of the  $n$ -point connected correlation function for noninteracting particles in real space, for  $0 < t_1 < \cdots < t_n$ ,

$$C_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = P(\mathbf{x}_1, t_1) \prod_{i=1}^n P(\mathbf{x}_{i+1}, t_{i+1} | \mathbf{x}_i, t_i) = P[\mathbf{x}(t_1) = \mathbf{x}_1, \dots, \mathbf{x}(t_n) = \mathbf{x}_n], \quad (\text{C5})$$

where  $P(\mathbf{x}_1, t_1) = N/V = \rho_0$ , and  $P(\mathbf{x}', t' | \mathbf{x}, t)$  is the propagator of a single particle, interpreted as the probability density of finding the particle at position  $\mathbf{x}'$  at time  $t'$ , given that it was at position  $\mathbf{x}$  at time  $t$ . The last equality holds for any Markovian stochastic process. In the case of Brownian motion, and assuming  $t' > t$ , the propagator takes the Gaussian form  $P(\mathbf{x}', t' | \mathbf{x}, t) = [4\pi D(t' - t)]^{-d/2} \exp\{-\frac{(\mathbf{x}' - \mathbf{x})^2}{4D(t' - t)}\}$ , and yields the real-space expression from Ref. [20].

#### APPENDIX D: DERIVATION OF THE THREE-POINT CORRELATION FUNCTIONS

In this Appendix, we give details on the derivation of Eq. (16) in the main text.

##### 1. Expression in Fourier space

The starting point of this derivation is the expansion of Eqs. (7) and (8) at order 2 in  $\lambda$ . They read

$$\partial_t q_2 = D\nabla^2 q_2 + \mu\rho_0 \nabla \cdot (\nabla V * q_2) - 2D\rho_0 \nabla^2 p_2 - \underbrace{2D\nabla \cdot (q_1 \nabla p_1) + \mu \nabla \cdot [q_1 (\nabla V * q_1)]}_{\equiv Q_2}, \quad (\text{D1})$$

$$\partial_t p_2 = -D\nabla^2 p_2 - \rho_0 \mu \nabla V * \nabla p_2 - \underbrace{D(\nabla p_1)^2 + \mu(\nabla p_1) \cdot (\nabla V * q_1) - \mu(q_1 \nabla p_1 * \nabla V)}_{\equiv P_2}, \quad (\text{D2})$$

where  $Q_2$  and  $P_2$  play the roles of sources. In Fourier space, the first equation yields the following expression of  $\tilde{q}_2$ :

$$\tilde{q}_2(\mathbf{k}, \omega) = \frac{2D\rho_0 k^2}{i\omega + \Omega(k)} \tilde{p}_2(\mathbf{k}, \omega) + \frac{\tilde{Q}_2(\mathbf{k}, \omega)}{i\omega + \Omega(k)}. \quad (\text{D3})$$



From Eq. (D2), one gets  $\tilde{p}_2(\mathbf{k}, \omega) = \tilde{P}_2(\mathbf{k}, \omega)/[i\omega - \Omega(\mathbf{k})]$ . Finally,  $\tilde{q}_2$  is given explicitly in terms of  $\tilde{P}_2$  and  $\tilde{Q}_2$ :

$$\tilde{q}_2(\mathbf{k}, \omega) = -\frac{2D\rho_0 k^2}{\Omega(\mathbf{k})^2 + \omega^2} \tilde{P}_2(\mathbf{k}, \omega) + \frac{\tilde{Q}_2(\mathbf{k}, \omega)}{i\omega + \Omega(\mathbf{k})}, \quad (\text{D4})$$

which are themselves explicit functions of the functions  $p_1$  and  $q_1$  determined explicitly at the previous step of the calculation. The goal of the calculation is to compute the three-point correlation functions:

$$\tilde{C}_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \omega_1, \omega_2, \omega_3) = \langle \tilde{\rho}(\mathbf{k}_1, \omega_1) \rho(\mathbf{k}_2, \omega_2) \rho(\mathbf{k}_3, \omega_3) \rangle_c, \quad (\text{D5})$$

$$= (2\pi)^{2(d+1)} \frac{\delta^2 q(\mathbf{k}_1, \omega_1)}{\delta \tilde{\lambda}(-\mathbf{k}_2, -\omega_2) \delta \tilde{\lambda}(-\mathbf{k}_3, -\omega_3)} \Big|_{\tilde{\lambda}=0}, \quad (\text{D6})$$

$$= (2\pi)^{2(d+1)} \frac{\delta^2 q_2(\mathbf{k}_1, \omega_1)}{\delta \tilde{\lambda}(-\mathbf{k}_2, -\omega_2) \delta \tilde{\lambda}(-\mathbf{k}_3, -\omega_3)}. \quad (\text{D7})$$

Using Eq. (D4), one gets

$$\begin{aligned} & \langle \tilde{\rho}(\mathbf{k}_1, \omega_1) \rho(\mathbf{k}_2, \omega_2) \rho(\mathbf{k}_3, \omega_3) \rangle_c \\ &= (2\pi)^{2(d+1)} \left\{ -\frac{2D\rho_0 k_1^2}{\Omega(\mathbf{k}_1)^2 + \omega_1^2} \frac{\delta^2 \tilde{P}_2(\mathbf{k}_1, \omega_1)}{\delta \tilde{\lambda}(-\mathbf{k}_2, -\omega_2) \delta \tilde{\lambda}(-\mathbf{k}_3, -\omega_3)} + \frac{1}{i\omega_1 + \Omega(\mathbf{k}_1)} \frac{\delta^2 \tilde{Q}_2(\mathbf{k}_1, \omega_1)}{\delta \tilde{\lambda}(-\mathbf{k}_2, -\omega_2) \delta \tilde{\lambda}(-\mathbf{k}_3, -\omega_3)} \right\}. \end{aligned} \quad (\text{D8})$$

From the definitions of  $\tilde{Q}_2$  and  $\tilde{P}_2$  [in Eqs. (D1) and (D2), respectively], one can compute their Fourier transforms as

$$\tilde{Q}_2(\mathbf{k}, \omega) = \int \frac{d\mathbf{k}' d\omega'}{(2\pi)^{d+1}} (\mathbf{k} - \mathbf{k}') \cdot \mathbf{k} \tilde{q}_1(\mathbf{k}', \omega') [2D\tilde{p}_1(\mathbf{k} - \mathbf{k}', \omega - \omega') - \mu \tilde{V}(\mathbf{k} - \mathbf{k}') \tilde{q}_1(\mathbf{k} - \mathbf{k}', \omega - \omega')], \quad (\text{D9})$$

$$\begin{aligned} \tilde{P}_2(\mathbf{k}, \omega) &= D \int \frac{d\mathbf{k}' d\omega'}{(2\pi)^{d+1}} \mathbf{k}' \cdot (\mathbf{k} - \mathbf{k}') \tilde{p}_1(\mathbf{k}', \omega') \tilde{p}_1(\mathbf{k} - \mathbf{k}', \omega - \omega') \\ &\quad - \mu \int \frac{d\mathbf{k}' d\omega'}{(2\pi)^{d+1}} \mathbf{k}' \cdot (\mathbf{k} - \mathbf{k}') \tilde{p}_1(\mathbf{k}', \omega') \tilde{V}(\mathbf{k} - \mathbf{k}') \tilde{q}_1(\mathbf{k} - \mathbf{k}', \omega - \omega') \\ &\quad + \mu \int \frac{d\mathbf{k}' d\omega'}{(2\pi)^{d+1}} \mathbf{k} \cdot (\mathbf{k} - \mathbf{k}') \tilde{q}_1(\mathbf{k}', \omega') \tilde{V}(\mathbf{k}) \tilde{p}_1(\mathbf{k} - \mathbf{k}', \omega - \omega'). \end{aligned} \quad (\text{D10})$$

Recalling for completeness the expressions of  $\tilde{p}_1$  and  $\tilde{q}_1$ :

$$\tilde{p}_1(\mathbf{k}, \omega) = \frac{\tilde{\lambda}(\mathbf{k}, \omega)}{\Omega(\mathbf{k}) - i\omega}, \quad (\text{D11})$$

$$\tilde{q}_1(\mathbf{k}, \omega) = \frac{2D\rho k^2 \tilde{\lambda}(\mathbf{k}, \omega)}{\Omega(\mathbf{k})^2 + \omega^2}, \quad (\text{D12})$$

one then computes the second functional derivatives of  $\tilde{Q}_2$  and  $\tilde{P}_2$ , which read

$$\begin{aligned} \frac{\delta^2 \tilde{Q}_2(\mathbf{k}_1, \omega_1)}{\delta \tilde{\lambda}(-\mathbf{k}_2, -\omega_2) \delta \tilde{\lambda}(-\mathbf{k}_3, -\omega_3)} &= \frac{1}{(2\pi)^{d+1}} \frac{2D\rho_0 k_2^2}{\Omega(\mathbf{k}_2)^2 + \omega_2^2} \mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \delta(\omega_1 + \omega_2 + \omega_3) \\ &\quad \times \left[ \frac{2D}{\Omega(\mathbf{k}_1 + \mathbf{k}_2) - i(\omega_1 + \omega_2)} - \mu \tilde{V}(\mathbf{k}_1 + \mathbf{k}_2) \frac{2D\rho_0 (\mathbf{k}_1 + \mathbf{k}_2)^2}{\Omega(\mathbf{k}_1 + \mathbf{k}_2)^2 + (\omega_1 + \omega_2)^2} \right] + 2 \leftrightarrow 3, \end{aligned} \quad (\text{D13})$$

and

$$\begin{aligned} \frac{\delta^2 \tilde{P}_2(\mathbf{k}_1, \omega_1)}{\delta \tilde{\lambda}(-\mathbf{k}_2, -\omega_2) \delta \tilde{\lambda}(-\mathbf{k}_3, -\omega_3)} &= \frac{1}{(2\pi)^{d+1}} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \delta(\omega_1 + \omega_2 + \omega_3) \left\{ \frac{D\mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{[\Omega(\mathbf{k}_2) + i\omega_2][\Omega(\mathbf{k}_1 + \mathbf{k}_2) - i(\omega_1 + \omega_2)]} \right. \\ &\quad + \mu \mathbf{k}_2 \cdot (\mathbf{k}_1 + \mathbf{k}_2) \tilde{V}(\mathbf{k}_1 + \mathbf{k}_2) \frac{2D\rho_0 (\mathbf{k}_1 + \mathbf{k}_2)^2}{[\Omega(\mathbf{k}_2) + i\omega_2][\Omega(\mathbf{k}_1 + \mathbf{k}_2)^2 + (\omega_1 + \omega_2)^2]} \\ &\quad \left. + \mu \mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2) \tilde{V}(\mathbf{k}_1) \frac{2D\rho_0 k_2^2}{[\Omega(\mathbf{k}_1 + \mathbf{k}_2) - i(\omega_1 + \omega_2)][\Omega(\mathbf{k}_2)^2 + \omega_2^2]} + 2 \leftrightarrow 3 \right\}. \end{aligned} \quad (\text{D14})$$

Finally, with Eq. (D8), one gets

$$\begin{aligned}
& \langle \tilde{\rho}(\mathbf{k}_1, \omega_1) \rho(\mathbf{k}_2, \omega_2) \rho(\mathbf{k}_3, \omega_3) \rangle_c \\
&= (2\pi)^{d+1} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \delta(\omega_1 + \omega_2 + \omega_3) \\
&\quad \times \left\{ \frac{2D^2 \rho_0(\mathbf{k}_1 + \mathbf{k}_2)}{\Omega(\mathbf{k}_1 + \mathbf{k}_2) - i(\omega_1 + \omega_2)} \cdot \left( \frac{k_1^2 k_2}{[\Omega(\mathbf{k}_2) + i\omega_2][\Omega(\mathbf{k}_1)^2 + \omega_1^2]} + \frac{2k_2^2 k_1}{[\Omega(\mathbf{k}_1) + i\omega_1][\Omega(\mathbf{k}_2)^2 + \omega_2^2]} \right) + 2 \leftrightarrow 3 \right. \\
&\quad - \mu(2D\rho_0)^2 \frac{\tilde{V}(\mathbf{k}_1 + \mathbf{k}_2)(\mathbf{k}_1 + \mathbf{k}_2)^2(\mathbf{k}_1 + \mathbf{k}_2)}{\Omega(\mathbf{k}_1 + \mathbf{k}_2)^2 + (\omega_1 + \omega_2)^2} \cdot \left( \frac{k_1^2 k_2}{[\Omega(\mathbf{k}_2) + i\omega_2][\Omega(\mathbf{k}_1)^2 + \omega_1^2]} + \frac{k_2^2 k_1}{[\Omega(\mathbf{k}_1) + i\omega_1][\Omega(\mathbf{k}_2)^2 + \omega_2^2]} \right) + 2 \leftrightarrow 3 \\
&\quad \left. - \mu(2D\rho_0)^2 \frac{\tilde{V}(\mathbf{k}_1)k_2^2 k_1^2 \mathbf{k}_1 \cdot (\mathbf{k}_1 + \mathbf{k}_2)}{[\Omega(\mathbf{k}_1 + \mathbf{k}_2) - i(\omega_1 + \omega_2)][\Omega(\mathbf{k}_1)^2 + \omega_1^2][\Omega(\mathbf{k}_2)^2 + \omega_2^2]} + 2 \leftrightarrow 3 \right\} \quad (D15)
\end{aligned}$$

$$\equiv \rho_0(2\pi)^{d+1} \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \delta(\omega_1 + \omega_2 + \omega_3) \psi_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \omega_1, \omega_2, \omega_3). \quad (D16)$$

The last equality defines  $\psi_3$ . This is the most general expression for the connected three-point correlation function. To simplify this expression, we introduce the shorthand notation  $\mathbf{K}_i \equiv (\mathbf{k}_i, \omega_i)$  and

$$\mathcal{G}(\mathbf{K}_i) = \frac{1}{\Omega(\mathbf{k}_i) + i\omega_i}. \quad (D17)$$

One gets the following expression for  $\psi_3$ :

$$\psi_3(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3) = \Gamma_3(\mathbf{K}_1, \mathbf{K}_2) \psi_2(\mathbf{K}_3) + \Gamma_3(\mathbf{K}_1, \mathbf{K}_3) \psi_2(\mathbf{K}_2) + \Gamma_3(\mathbf{K}_2, \mathbf{K}_3) \psi_2(\mathbf{K}_1), \quad (D18)$$

where  $\psi_2(\mathbf{K})$  is a shorthand for  $\psi_2(-\mathbf{K}, \mathbf{K})$ , and  $\Gamma_3(\mathbf{K}, \mathbf{K}')$  reads

$$\Gamma_3(\mathbf{K}, \mathbf{K}') \equiv -2D \mathbf{k} \cdot \mathbf{k}' \mathcal{G}(\mathbf{K}) \mathcal{G}(\mathbf{K}') + \mu \mathbf{k} \cdot \mathbf{k}' [\mathcal{G}(\mathbf{K}) \tilde{V}(\mathbf{K}') \psi_2(\mathbf{K}') + \mathbf{K} \leftrightarrow \mathbf{K}'] \quad (D19)$$

$$= \psi_2(\mathbf{K}, \mathbf{K}') + \mu \mathbf{k} \cdot \mathbf{k}' [\mathcal{G}(\mathbf{K}) \tilde{V}(\mathbf{K}') \psi_2(\mathbf{K}') + \mathbf{K} \leftrightarrow \mathbf{K}']. \quad (D20)$$

As written, (D18), the expression of the three-point correlation function is explicitly symmetric under the exchange  $\mathbf{K}_i \leftrightarrow \mathbf{K}_j$ .

## 2. Expression in real time

We next invert the Fourier transforms with respect to time in order to compute

$$\langle \tilde{\rho}(\mathbf{k}_1, t_1) \tilde{\rho}(\mathbf{k}_2, t_2) \tilde{\rho}(\mathbf{k}_3, t_3) \rangle_c = \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} e^{i\omega_1 t_1} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} e^{i\omega_2 t_2} \int_{-\infty}^{\infty} \frac{d\omega_3}{2\pi} e^{i\omega_3 t_3} \langle \tilde{\rho}(\mathbf{k}_1, \omega_1) \tilde{\rho}(\mathbf{k}_2, \omega_2) \tilde{\rho}(\mathbf{k}_3, \omega_3) \rangle_c. \quad (D21)$$

Using the fact that the correlation function is in fact proportional to  $\delta(\omega_1 + \omega_2 + \omega_3)$ , we get, using the definition of  $\psi_3$  [Eq. (D16)],

$$\langle \tilde{\rho}(\mathbf{k}_1, t_1) \tilde{\rho}(\mathbf{k}_2, t_2) \tilde{\rho}(\mathbf{k}_3, t_3) \rangle_c = (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} e^{i\omega_1(t_1 - t_3)} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} e^{i\omega_2(t_2 - t_3)} \psi_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \omega_1, \omega_2, -\omega_1 - \omega_2). \quad (D22)$$

One can get a general expression without making any assumption of the order of the times  $t_1, t_2$ , and  $t_3$ , i.e., by defining quantities like  $\sigma_{ab} = \text{sgn}(t_a - t_b)$ , and expressing the final result in terms of  $\sigma_{ab}$ . To simplify, we make the choice  $0 < t_1 < t_2 < t_3$ , and the result reads

$$\begin{aligned}
\langle \tilde{\rho}(\mathbf{k}_1, t_1) \tilde{\rho}(\mathbf{k}_2, t_2) \tilde{\rho}(\mathbf{k}_3, t_3) \rangle_c &= \rho_0 D \left\{ \left[ \frac{2Dk_2^2 k_3 \cdot (\mathbf{k}_1 v_1 + \mathbf{k}_2 v_2)}{\Omega_1 + \Omega_2 - \Omega_3} - 2\mathbf{k}_2 \cdot \mathbf{k}_3 \right] \frac{e^{-(t_2 - t_1)\Omega_1 - (t_3 - t_2)\Omega_3}}{(\Omega_2 - \Omega_1 + \Omega_3)(1 + v_1)} \right. \\
&\quad + \frac{\mathbf{k}_3 \cdot (\mathbf{k}_1 v_1 + \mathbf{k}_2 v_2)}{(1 + v_1)(1 + v_2)} \frac{e^{-(t_3 - t_1)\Omega_1 - (t_3 - t_2)\Omega_2}}{\Omega_3 - \Omega_2 - \Omega_1} + \frac{\mathbf{k}_1 \cdot (\mathbf{k}_2 v_2 + \mathbf{k}_3 v_3)}{(1 + v_2)(1 + v_3)} \frac{e^{-(t_2 - t_1)\Omega_2 - (t_3 - t_1)\Omega_3}}{\Omega_1 - \Omega_2 - \Omega_3} \\
&\quad \left. + \frac{\mathbf{k}_2 \cdot (\mathbf{k}_1 v_1 + \mathbf{k}_3 v_3)}{(1 + v_1)(1 + v_3)} \frac{e^{-(t_2 - t_1)\Omega_1 - (t_3 - t_2)\Omega_3}}{\Omega_2 - \Omega_1 + \Omega_3} \right\},
\end{aligned}$$

where we use the shorthand notations  $v_j = \frac{\rho_0 \tilde{V}(\mathbf{k}_j)}{k_B T}$  and  $\Omega_j = Dk_j^2(1 + \frac{\rho_0 \tilde{V}(\mathbf{k}_j)}{k_B T})$ . Finally, taking  $t_2 = t_3 = t > 0$  and  $t_1 = 0$ , we get Eq. (16) in the main text.

### APPENDIX E: EXPRESSION OF THE FOUR-POINT CORRELATION FUNCTION

In this Appendix, we give details on the expression of the four-point correlation function  $\tilde{C}_4(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4)$  whose expression is provided in the variables  $(\mathbf{k}, t)$  with times ordered as  $0 < t_1 < t_2 < t_3 < t_4$  in the associated Python notebook [59]. To this end, we first recall the expressions of the two- and three-point correlation functions, and then we show how they relate to the four-point correlation function.

First, we recall the definition of the amplitude  $\psi_n$  associated with the  $n$ -point correlation function  $\tilde{C}_n$  [see Eq. (10) in the main text]:

$$\tilde{C}_n(\mathbf{K}_1, \dots, \mathbf{K}_n) = \rho_0 (2\pi)^{d+1} \psi_n(\mathbf{K}_1, \dots, \mathbf{K}_n) \delta\left(\sum_i \mathbf{K}_i\right), \quad (\text{E1})$$

such that the amplitude  $\psi_2$  associated with the two-point correlation function  $\tilde{C}_2$ , given by Eq. (11) in the main text, reads

$$\psi_2(\mathbf{K}_1, \mathbf{K}_2) = -2D \mathbf{k}_1 \cdot \mathbf{k}_2 \mathcal{G}(\mathbf{K}_1) \mathcal{G}(\mathbf{K}_2). \quad (\text{E2})$$

With this expression,  $\psi_3$ , given by Eq. (D18), can be written as a function of  $\psi_2$ :

$$\begin{aligned} \psi_3(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3) &= \Gamma_3(\mathbf{K}_1, \mathbf{K}_2) \psi_2(\mathbf{K}_3) + \Gamma_3(\mathbf{K}_1, \mathbf{K}_3) \psi_2(\mathbf{K}_2) \\ &\quad + \Gamma_3(\mathbf{K}_2, \mathbf{K}_3) \psi_2(\mathbf{K}_1). \end{aligned} \quad (\text{E3})$$

Here,  $\psi_2(\mathbf{K})$  denotes  $\psi_2(-\mathbf{K}, \mathbf{K})$ , and the function  $\Gamma_3(\mathbf{K}, \mathbf{K}')$  is given by

$$\begin{aligned} \Gamma_3(\mathbf{K}, \mathbf{K}') &= \psi_2(\mathbf{K}, \mathbf{K}') + \mu \mathbf{k} \cdot \mathbf{k}' [\mathcal{G}(\mathbf{K}) \tilde{V}(\mathbf{k}') \psi_2(\mathbf{K}') \\ &\quad + \mathbf{K} \leftrightarrow \mathbf{K}']. \end{aligned} \quad (\text{E4})$$

Finally, following the same steps as in Appendix D and expanding up to third order in  $\lambda$ , one obtains the four-point correlation function. Although the calculation is cumbersome, the result reads

$$\begin{aligned} \psi_4(\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4) &= \Gamma_4(\mathbf{K}_1, \mathbf{K}_4) \psi_3(\mathbf{K}_1 + \mathbf{K}_4, \mathbf{K}_2, \mathbf{K}_3) \\ &\quad + \Gamma_4(\mathbf{K}_2, \mathbf{K}_3) \{ \Gamma_2(\mathbf{K}_1, \mathbf{K}_2 + \mathbf{K}_3) \psi_2(\mathbf{K}_4) \\ &\quad + \Gamma_2(\mathbf{K}_4, \mathbf{K}_2 + \mathbf{K}_3) \psi_2(\mathbf{K}_1) \\ &\quad + \mu \mathbf{k}_1 \cdot (\mathbf{k}_2 + \mathbf{k}_3) \tilde{V}(\mathbf{k}_1) \mathcal{G}(\mathbf{K}_2 + \mathbf{K}_3) \psi_2(\mathbf{K}_1) \psi_2(\mathbf{K}_4) \\ &\quad + \mu \mathbf{k}_4 \cdot (\mathbf{k}_2 + \mathbf{k}_3) \tilde{V}(\mathbf{k}_4) \mathcal{G}(\mathbf{K}_2 + \mathbf{K}_3) \psi_2(\mathbf{K}_1) \psi_2(\mathbf{K}_4) \} \\ &\quad + 4 \leftrightarrow 2 + 4 \leftrightarrow 3, \end{aligned} \quad (\text{E5})$$

where the function  $\Gamma_4$  is defined as

$$\begin{aligned} \Gamma_4(\mathbf{K}, \mathbf{K}') &\equiv \Gamma_3(\mathbf{K}, \mathbf{K}') - \mu \tilde{V}(\mathbf{k} + \mathbf{k}') (\mathbf{k} + \mathbf{k}') \\ &\quad \cdot [\mathbf{k} \mathcal{G}(\mathbf{K}) \psi_2(\mathbf{K}') + \mathbf{K} \leftrightarrow \mathbf{K}']. \end{aligned} \quad (\text{E6})$$

As written, the four-point correlation function (E5) is not explicitly symmetric under the exchange  $\mathbf{K}_i \leftrightarrow \mathbf{K}_j$ , but we have verified this symmetry numerically. Furthermore, the Fourier transform with respect to  $\omega$  can be inverted, and the expression of the four-point correlation function  $C_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4; t_1, t_2, t_3, t_4)$ , with times ordered as

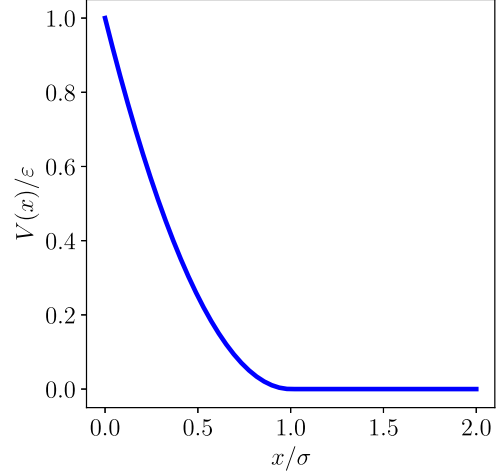


FIG. 3. Plot of the interaction potential used in the Brownian dynamics simulations:  $V(x) = \varepsilon(1 - x/\sigma)^2 \Theta(\sigma - x)$ .

$0 < t_1 < t_2 < t_3 < t_4$ , is provided in the associated Python notebook [59].

### APPENDIX F: NUMERICAL SIMULATIONS

We performed Brownian dynamics simulations, i.e., time-integration of the overdamped Langevin equation given as Eq. (1). For a one-dimensional system, the integrated equation reads

$$x_\alpha(t + \delta t) = x_\alpha(t) - \mu \delta t \sum_{\beta \neq \alpha} V'(|x_\alpha - x_\beta|) + \sqrt{2D\delta t} Z_{0,1}, \quad (\text{F1})$$

where  $Z_{0,1}$  is a random number drawn from a normal distribution of mean 0 and variance 1. The  $N = 100$  particles are initially placed uniformly on a line of length  $L$  (which is chosen to attain the chosen density  $\rho_0 = N/L$ ). The particles interact via a harmonic repulsive potential, which reads  $V(x) = \varepsilon(1 - x/\sigma)^2 \Theta(\sigma - x)$ , where  $\Theta$  is the Heaviside function (see Fig. 3 for a plot of the interaction potential). Its derivative, which is needed to compute the forces in Eq. (F1), reads  $V'(x) = -2\varepsilon(1 - x/\sigma) \Theta(\sigma - x)$ . The length  $\sigma$  is taken equal to 1 and sets the unit length of the simulation. We also take  $D = \mu = 1$  (this sets the typical diffusive timescale  $\sigma^2/D$  to 1), and  $\varepsilon = 0.5k_B T$ . The integration time step is  $\delta t = 0.01$ .

The simulation is initially run for  $10^4$  steps, for equilibration. The correlation functions shown in the main text are computed on the next  $10^4$  steps: at each time step, the Fourier transform of the microscopic density is computed as  $\tilde{\rho}(k, t) = \sum_{\alpha=1}^N e^{-ikx_\alpha(t)}$ . Importantly, since the system has periodic boundary conditions, the wave vectors  $k$  must be chosen as multiples of  $2\pi/L$  [66]. The structure factors and correlation functions are computed directly from their definition in the main text in terms of the functions  $\tilde{\rho}(k, t)$ , and averages are performed over initial conditions and noise realisations (typically  $2 \times 10^4$  independent realizations).

Finally, we emphasize that analytical calculations are performed in the thermodynamic limit where  $N, L \rightarrow \infty$  with a fixed density  $\rho_0 = N/L$ . Results for finite-size systems, which are required to make comparisons with numerical simulations, can be obtained by making the change  $\delta(k) \rightarrow [\mathcal{V}/(2\pi)]\delta_{k,0}$ .

- [1] D. A. McQuarrie, *Statistical Mechanics* (Harper and Row, New York, 1976).
- [2] M. Doi and S. F. Edwards, *The Theory of Polymer Dynamics* (Oxford University Press, 1988).
- [3] J. P. Hansen and I. R. McDonald, *Theory of Simple Liquids*, 3rd ed. (Academic Press, New York, 2005).
- [4] S. W. Lovesey, *Theory of Neutron Scattering from Condensed Matter* (Wiley Interscience, New York, 1986), Vol. I.
- [5] B. J. Berne and R. Pecora, *Dynamic Light Scattering* (Dover, 2000).
- [6] N. Stribeck, *X-Ray Scattering of Soft Matter*, Springer Laboratory (Springer, Berlin, 2007).
- [7] P. Lindner and J. Oberdisse, *Neutrons, X-rays, and Light: Scattering Methods Applied to Soft Condensed Matter*, 2nd ed. (Elsevier, Amsterdam, 2024).
- [8] B. J. Alder, Triplet correlations in hard spheres, *Phys. Rev. Lett.* **12**, 317 (1964).
- [9] A. Rahman, Triplet correlations in liquids, *Phys. Rev. Lett.* **12**, 575 (1964).
- [10] P. A. Egelstaff, D. I. Page, and C. R. T. Heard, Experimental study of the triplet correlation function for simple liquids, *J. Phys. C* **4**, 1453 (1971).
- [11] A. D. J. Haymet, S. A. Rice, and W. G. Madden, Triplet correlations in the Lennard-Jones fluid, *J. Chem. Phys.* **75**, 4696 (1981).
- [12] J. Barrat, J. Hansen, and G. Pastore, On the equilibrium structure of dense fluids: Triplet correlations, integral equations and freezing, *Mol. Phys.* **63**, 747 (1988).
- [13] P. Attard, Spherically inhomogeneous fluids. I. Percus–Yevick hard spheres: Osmotic coefficients and triplet correlations, *J. Chem. Phys.* **91**, 3072 (1989).
- [14] D. Coslovich, Static triplet correlations in glass-forming liquids: A molecular dynamics study, *J. Chem. Phys.* **138**, 12A539 (2013).
- [15] I. Pihlajamaa and L. M. C. Janssen, Comparison of integral equation theories of the liquid state, *Phys. Rev. E* **110**, 044608 (2024).
- [16] W. Götze, *Complex Dynamics of Glass-Forming Liquids* (Oxford University Press, 2009).
- [17] R. van Zon and J. Schofield, Mode-coupling theory for multiple-point and multiple-time correlation functions, *Phys. Rev. E* **65**, 011106 (2001).
- [18] K. Kawasaki, Stochastic model of slow dynamics in supercooled liquids and dense colloidal suspensions, *Physica A* **208**, 35 (1994).
- [19] D. S. Dean, Langevin equation for the density of a system of interacting Langevin processes, *J. Phys. A* **29**, L613 (1996).
- [20] A. Velenich, C. Chamon, L. F. Cugliandolo, and D. Kreimer, On the Brownian gas: A field theory with a Poissonian ground state, *J. Phys. A* **41**, 235002 (2008).
- [21] B. Kim, K. Kawasaki, H. Jacquin, and F. van Wijland, Equilibrium dynamics of the Dean-Kawasaki equation: Mode-coupling theory and its extension, *Phys. Rev. E* **89**, 012150 (2014).
- [22] P.-H. Chavanis, Hamiltonian and Brownian systems with long-range interactions: V. Stochastic kinetic equations and theory of fluctuations, *Physica A* **387**, 5716 (2008).
- [23] V. Démery, O. Bénichou, and H. Jacquin, Generalized Langevin equations for a driven tracer in dense soft colloids: Construction and applications, *New J. Phys.* **16**, 053032 (2014).
- [24] D. S. Dean and R. Podgornik, Relaxation of the thermal Casimir force between net neutral plates containing Brownian charges, *Phys. Rev. E* **89**, 032117 (2014).
- [25] A. Poncet, O. Bénichou, V. Démery, and D. Nishiguchi, Pair correlation of dilute active Brownian particles: From low-activity dipolar correction to high-activity algebraic depletion wings, *Phys. Rev. E* **103**, 012605 (2021).
- [26] A. Poncet, O. Bénichou, V. Démery, and G. Oshanin, Universal long ranged correlations in driven binary mixtures, *Phys. Rev. Lett.* **118**, 118002 (2017).
- [27] V. Démery and D. S. Dean, The conductivity of strong electrolytes from stochastic density functional theory, *J. Stat. Mech.* (2016) 023106.
- [28] S. Mahdisoltani and R. Golestanian, Long-range fluctuation-induced forces in driven electrolytes, *Phys. Rev. Lett.* **126**, 158002 (2021).
- [29] D. Martin, C. Nardini, M. E. Cates, and É. Fodor, Extracting maximum power from active colloidal heat engines, *Europhys. Lett.* **121**, 60005 (2018).
- [30] T. H. N. Minh, B. Rotenberg, and S. Marbach, Ionic fluctuations in finite volumes: Fractional noise and hyperuniformity, *Faraday Discuss.* **246**, 225 (2023).
- [31] F. Bouchet, K. Gawedzki, and C. Nardini, Perturbative calculation of quasi-potential in non-equilibrium diffusions: A mean-field example, *J. Stat. Phys.* **163**, 1157 (2016).
- [32] P. Illien, The Dean-Kawasaki equation and stochastic density functional theory, *Rep. Prog. Phys.* **88**, 086601 (2025).
- [33] Throughout the paper, the convention for Fourier transformation is as follows:
$$\tilde{f}(\mathbf{k}, \omega) = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} \int dt e^{-i\omega t} f(\mathbf{r}, t),$$
and its inverse,
$$f(\mathbf{r}, t) = \frac{1}{(2\pi)^d} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \int d\omega e^{i\omega t} \tilde{f}(\mathbf{k}, \omega).$$
- [34] P. C. Martin, E. D. Siggia, and H. A. Rose, Statistical dynamics of classical systems, *Phys. Rev. A* **8**, 423 (1973).
- [35] H.-K. Janssen, On a Lagrangean for classical field dynamics and renormalization group calculations of dynamical critical properties, *Z. Phys. B* **23**, 377 (1976).
- [36] P. L. Krapivsky, K. Mallick, and T. Sadhu, Large deviations in single-file diffusion, *Phys. Rev. Lett.* **113**, 078101 (2014).
- [37] P. L. Krapivsky, K. Mallick, and T. Sadhu, Tagged particle in single-file diffusion, *J. Stat. Phys.* **160**, 885 (2015).
- [38] K. Mallick, H. Moriya, and T. Sasamoto, Exact solution of the macroscopic fluctuation theory for the symmetric exclusion process, *Phys. Rev. Lett.* **129**, 040601 (2022).
- [39] R. Dandekar, P. L. Krapivsky, and K. Mallick, Dynamical fluctuations in the Riesz gas, *Phys. Rev. E* **107**, 044129 (2023).
- [40] R. Dandekar, P. L. Krapivsky, and K. Mallick, Current fluctuations in the Dyson gas, *Phys. Rev. E* **110**, 064153 (2024).
- [41] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1985).
- [42] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
- [43] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 4th ed. (Oxford University Press, 2002).
- [44] A. Andreanov, G. Biroli, and A. Lefèvre, Dynamical field theory for glass-forming liquids, self-consistent resummations and time-reversal symmetry, *J. Stat. Mech.* (2006) P07008.

- [45] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Fluctuations in stationary nonequilibrium states of irreversible processes, *Phys. Rev. Lett.* **87**, 040601 (2001).
- [46] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory for stationary nonequilibrium states, *J. Stat. Phys.* **107**, 635 (2002).
- [47] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Macroscopic fluctuation theory, *Rev. Mod. Phys.* **87**, 593 (2015).
- [48] B. Derrida and A. Gerschenfeld, Current fluctuations in one dimensional diffusive systems with a step initial density profile, *J. Stat. Phys.* **137**, 978 (2009).
- [49] P. L. Krapivsky and B. Meerson, Fluctuations of current in nonstationary diffusive lattice gases, *Phys. Rev. E* **86**, 031106 (2012).
- [50] A. Grabsch, A. Poncet, P. Rizkallah, P. Illien, and O. Bénichou, Exact closure and solution for spatial correlations in single-file diffusion, *Sci. Adv.* **8**, eabm5043 (2022).
- [51] A. Grabsch, P. Rizkallah, A. Poncet, P. Illien, and O. Bénichou, Exact spatial correlations in single-file diffusion, *Phys. Rev. E* **107**, 044131 (2023).
- [52] T. Berlioz, O. Bénichou, and A. Grabsch, Tracer and current fluctuations in driven diffusive systems, *Phys. Rev. Lett.* **134**, 247101 (2025).
- [53] A. Poncet, O. Bénichou, and P. Illien, Cumulant generating functions of a tracer in quenched dense symmetric exclusion processes, *Phys. Rev. E* **103**, L040103 (2021).
- [54] Analytical calculations are performed in the thermodynamic limit where  $N, \mathcal{V} \rightarrow \infty$  with a fixed density  $\rho_0 = N/\mathcal{V}$ . Results for finite-size systems, which are required to make comparisons with numerical simulations, can be obtained by making the change  $\delta(\mathbf{k}) \rightarrow [\mathcal{V}/(2\pi)^d]\delta_{\mathbf{k},0}$ .
- [55] A. A. Louis, P. G. Bolhuis, and J. P. Hansen, Mean-field fluid behavior of the Gaussian core model, *Phys. Rev. E* **62**, 7961 (2000).
- [56] A. A. Louis, P. G. Bolhuis, J. P. Hansen, and E. J. Meijer, Can polymer coils be modeled as “soft colloids”? *Phys. Rev. Lett.* **85**, 2522 (2000).
- [57] H. W. Jackson and E. Feenberg, Energy spectrum of elementary excitations in helium II, *Rev. Mod. Phys.* **34**, 686 (1962).
- [58] S. Ichimaru, Dielectric response function of electron liquids, *Phys. Rev. A* **2**, 494 (1970).
- [59] P. Illien, A. Carof, and L. Le Bon, Data and codes associated with the publication ‘Non-Gaussian density fluctuations in the Dean-Kawasaki equation’, Recherche Data Gouv, V1 (2025), <https://doi.org/10.57745/NUBD3G>.
- [60] H. Bonneau, V. Démery, and É. Raphaël, Temporal response of the conductivity of electrolytes, *J. Stat. Mech.* (2023) 073205.
- [61] Y. Avni, R. M. Adar, D. Andelman, and H. Orland, Conductivity of concentrated electrolytes, *Phys. Rev. Lett.* **128**, 098002 (2022).
- [62] O. Bernard, M. Jardat, B. Rotenberg, and P. Illien, On analytical theories for conductivity and self-diffusion in concentrated electrolytes, *J. Chem. Phys.* **159**, 164105 (2023).
- [63] P. Illien, A. Carof, and B. Rotenberg, Stochastic density functional theory for ions in a polar solvent, *Phys. Rev. Lett.* **133**, 268002 (2024).
- [64] D. Lesnicki, C. Y. Gao, B. Rotenberg, and D. T. Limmer, Field-dependent ionic conductivities from generalized fluctuation-dissipation relations, *Phys. Rev. Lett.* **124**, 206001 (2020).
- [65] D. Lesnicki, C. Y. Gao, D. T. Limmer, and B. Rotenberg, On the molecular correlations that result in field-dependent conductivities in electrolyte solutions, *J. Chem. Phys.* **155**, 014507 (2021).
- [66] G. R. Kneller, V. Keiner, M. Kneller, and M. Schiller, *nMOLDYN*: A program package for a neutron scattering oriented analysis of molecular dynamics simulations, *Comput. Phys. Commun.* **91**, 191 (1995).